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Totally Positive Functions and Totally Bounded Functions on $[-1, 1]$

ALAN L. HORWITZ

Department of Mathematics, Pennsylvania State University, Delaware County Campus, 25 Yearsly Mill Road, Media, Pennsylvania 19063, U.S.A.

AND

LEE A. RUBEL*

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, Illinois 61801, U.S.A.

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A function f on $[-1, 1]$ is said to be totally positive if all its Lagrange interpolants are positive on $[-1, 1]$. It is said to be totally bounded if there is a uniform bound on all its Lagrange interpolants on $[-1, 1]$. These classes of functions are studied here. © 1988 Academic Press, Inc.

INTRODUCTION

This paper continues our project on inverse interpolation begun in [HR1]—our general task being to deduce some property of a function f from some property or properties of its set $\mathscr{L}(f)$ of Lagrange interpolants. In this paper our two properties are:

(1) Uniform boundedness in the sup norm on $[-1, 1]$.

(2) Positivity on $[-1, 1]$.

The first condition will be shown to imply that f is analytic in a certain region E containing $[-1, 1]$, while the second implies infinite differentiability on $[-1, 1]$. Before we give more precise definitions, we need some preliminaries.

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If f is a real-valued function on a set S, we say that a polynomial p of degree n is a Lagrange interpolant of f if there are $n+1$ distinct numbers $\{x_0, x_1, ..., x_n\} \subseteq S$ such that $p(x_j) = f(x_j)$ for $j = 0, 1, ..., n$. (Of course there may be other points of agreement as well.) We find it most convenient to use the Newton form for the interpolating polynomial:

$$p(x) = f(x_0) + f[x_1, x_0](x - x_0) + \cdots + f[x_n, ..., x_1, x_0](x - x_0) \cdots (x - x_{n-1}).$$

We use the notation $p(x) = L(f; x_0, ..., x_n)$, where $f[x_j, ..., x_0]$ is defined inductively by

$$f[x_j, ..., x_0] \equiv \frac{f[x_j, ..., x_1] - f[x_{j-1}, ..., x_0]}{x_j - x_0}.$$

(This is just the well-known jth-order divided-difference of f.) We also make use of the error formula (see [IK])

$$E(x) \equiv f(x) - p(x) = (x - x_0) \cdots (x - x_n) f[x, x_n, ..., x_0].$$

The set of *all* Lagrange interpolants of f is denoted by $\mathscr{L}(f)$.

DEFINITION 1. A real-valued function f defined on S is said to be totally bounded on S if there exists an M such that $|p(x)| \leqslant M$ for all $p \in \mathscr{L}(f)$ and all $x \in S$. We write

$$\|f\|_{\text{TBS}} = \sup_{\substack{x \in S \\ p \in \mathscr{L}(f)}} |p(x)| \tag{1}$$

and denote the class of all such functions by TBS.

Most of this paper will focus on TBI, $I = [-1, 1]$, and in that case $\|\cdot\|_{\text{TBI}}$ gives a norm and TBI is a normed linear space. We shall see soon that TBI is in fact a Banach space.

DEFINITION 2. A real-valued function f defined on S is said to be totally positive if $p(x) > 0$ $\forall x \in S$ and $\forall p \in \mathscr{L}(f)$. We denote the class of such functions by TPS.

Again, our main focus will be on TPI. At this point it is natural to ask: Are there any non-polynomials in TPI? In fact, are there any non-linear functions in TPI?

The answer is yes, and we now indicate why. It is easily seen that TBI $\supseteq$ {polynomials}, and since TBI is a Banach space, there must be non-polynomials f in TBI (this can also be shown directly)—this follows from the Baire Category Theorem. But then $f(x)+M \in \mathrm{TPI}$ for sufficiently large M.

For the unbounded interval $[0, \infty)$, however, it turns out that $\mathrm{TP}[0, \infty)$ consists of *linear functions only*!

Remarks. (1) If some of the points of interpolation coalesce, we get Hermite interpolation. In particular, if $x_0 = \cdots = x_n$ we get the nth-order Taylor interpolant at x_0:

$$s_n(x; x_0) \equiv f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

By taking limits, if $f \in \mathrm{TBI}$, then $|s_n(x; x_0)| \leqslant \|f\|_{\mathrm{TBI}}$ for any $x, x_0 \in I$, $n \geqslant 0$.

(2) To be consistent we have defined TPS for interpolation at *distinct points*—if we allow coalescing nodes then the interpolant could vanish on S.

1. Totally Bounded Functions on I

THEOREM 1. *The norm defined above makes* TBI *a Banach space.*

Proof. Suppose $\{f_n\}$ is a Cauchy sequence in TBI. Then clearly $\{f_n\}$ is Cauchy in the uniform norm on $[-1, 1]$ (just consider constant interpolants), and hence there exists $f \in C[-1, 1]$ such that $f_n \to f$ uniformly on $[-1, 1]$. First we claim that

$$f \in \mathrm{TBI}. \tag{2}$$

To prove (2), let L be any Lagrange interpolation operator. Since a Cauchy sequence in a normed space is bounded, $\exists M$ (independent of n, L, and x) such that $|L(f_n(x))| \leqslant M$. Then

$$\begin{aligned} L(f)(x) &= L(f-f_n)(x) + L(f_n(x)) \\ &\Rightarrow |L(f)(x)| \leqslant |L(f-f_n)(x)| + M. \end{aligned}$$

But for any fixed L and x, $L(f_n)(x) \to L(f)(x)$ and hence we can force $|L(f)(x)| \leqslant 1 + M$, say, for any L, $x \in [-1, 1]$. This proves (2). Now we claim

$$f_n \to f \qquad \text{in the TBI norm.} \tag{3}$$

Now for any $x \in [-1, 1]$, and any L,

$$|L(f-f_n)(x)| \leqslant |L(f-f_m)(x)| + |L(f_m - f_n)(x)|.$$

For $\varepsilon > 0$ given, we can choose N_1 (depending on x, L) such that $|L(f-f_m)(x)| < \varepsilon/2$, $\forall m \geq N_1$. Also, we can choose N_2 (independent of x, L) such that $|L(f_m - f_n)(x)| < \varepsilon/2$, $\forall m$, $n \geq N_2$, since $\{f_n\}$ is Cauchy in TBI. Then for $n \geq N_2$, $|L(f-f_n)(x)| < \varepsilon$.

Remark. In proving Theorem 1 we only need that $\|f\|_{\infty(I)} \leq \|f\|_{\text{TBI}}$, and that if $f_n \to f$ uniformly on I, $L(f_n) \to L(f)$ pointwise on I, for any Lagrange interpolation operator L.

THEOREM 2. *Let f be defined on $[-1, 1]$ and suppose that there exists a real number r such that $p(x) \geq r$, $\forall x \in [-1, 1]$ and $\forall p \in \mathscr{L}(f)$. Then $f \in C^\infty[-1, 1]$.*

Proof. First we have

$$f \text{ is } \textit{bounded} \text{ on } [-1, 1]. \tag{4}$$

To prove (4), suppose (taking subsequences if necessary) that $f(x^{(j)}) \to +\infty$ with $x^{(j)} \to c \in [-1, 1]$. Consider $P_j(x) = f(x_0) + (x - x_0) f[x^{(j)}, x_0]$, the linear interpolant to f at $\{x_0, x^{(j)}\}$. If $c \neq -1$, choose x_0 such that $-1 < x_0 < c$. Then $p_j(-1) \to -\infty$, a contradiction. (If $c = -1$, choose x_0 such that $-1 < x_0 < 1$. Then $p_j(1) \to -\infty$.) Note that $f(x^{(j)})$ cannot tend to $-\infty$, since the same would be true for the constant interpolants. Now we make the following *inductive* hypothesis:

$$|f[x_0, ..., x_n]| \leq M_n \qquad \text{for all choices of points}$$
$$-1 \leq x_0 < x_1 < \cdots < x_n \leq 1. \tag{5}$$

Suppose $|f[x_0^{(j)}, ..., x_{n+1}^{(j)}]| \to +\infty$ for some sequence $\{x^{(j)}\}$, $x^{(j)} = (x_0^{(j)}, ..., x_{n+1}^{(j)}) \in I^{n+2}$, with all coordinates distinct. Again taking subsequences if necessary, assume $\{x^{(j)}\} \to x = (x_0, ..., x_{n+1})$.

Now choose some point $x_{n+2} \in (-1, 1) \setminus \{x_0, ..., x_{n+1}\}$. Consider

$$\begin{aligned} p_j(x) &\equiv L(f; x_0^{(j)}, ..., x_{n+1}^{(j)}, x_{n+2}) \\ &= f(x_0^{(j)}) + \cdots + (x - x_0^{(j)}) \cdots (x - x_n^{(j)}) f[x_0^{(j)}, ..., x_n^{(j)}, x_{n+2}] \\ &\quad + (x - x_0^{(j)}) \cdots (x - x_n^{(j)})(x - x_{n+2}) f[x_0^{(j)}, ..., x_n^{(j)}, x_{n+2}, x_{n+1}^{(j)}]. \end{aligned}$$

(Note that

$$f[x_0^{(j)}, ..., x_n^{(j)}, x_{n+2}] \equiv \frac{f[x_0^{(j)}, ..., x_n^{(j)}] - f[x_1^{(j)}, ..., x_{n+2}]}{x_0^{(j)} - x_{n+2}},$$

which is defined for large j since $x_0^{(j)}$ stays away from x_{n+2}, remains *bounded* by (5).) Now

$$\begin{aligned} f[x_0^{(j)}, ..., x_n^{(j)}, x_{n+2}, x_{n+1}^{(j)}] &= f[x_0^{(j)}, ..., x_n^{(j)}, x_{n+1}^{(j)}, x_{n+2}] \\ &\equiv \frac{f[x_0^{(j)}, ..., x_{n+1}^{(j)}] - f[x_1^{(j)}, ..., x_{n+2}]}{x_0^{(j)} - x_{n+2}} \to +\infty \text{ or } -\infty \end{aligned}$$

for some *subsequence* since $f[x_1^{(j)}, ..., x_{n+2}]$ remains bounded. Then we just choose $a \in [-1, 1]$ so that $(a - x_0^{(j)}) \cdots (a - x_n^{(j)})(a - x_{n+2})$ has the *opposite sign* from $f[x_0^{(j)}, ..., x_{n+1}^{(j)}, x_{n+2}]$. Then $p_j(a) \to -\infty$, which is a contradiction.

Hence we have that $|f[x_0, ..., x_{n+1}]| \leqslant M_{n+1}$ for all points x_j such that $-1 \leqslant x_0 < \cdots < x_{n+1} \leqslant 1$. So by *induction* (using (4) to get started), for each positive integer n, $|f[x_0, ..., x_n]| \leqslant M_n$. Then for sufficiently large c (depending on n), $g(x) = f(x) + ce^x$ satisfies

$$g[x_0, ..., x_n] > 0 \qquad \text{for all } x_j \text{ such that } -1 \leqslant x_0 < \cdots < x_n \leqslant 1. \tag{6}$$

Now we should also note that

$$f \in C[-1, 1] \qquad \text{(and thus } g \in C[-1, 1] \text{ also).} \tag{7}$$

Indeed, $|f[x, y]| \leqslant M_1$ for all $x \neq y$ in $[-1, 1]$, and (7) follows immediately. Then by [BW], the derivative $g^{(n-2)}$ *exists* in $(-1, 1)$, for $n > 2$.

Now take $n = 4$. By choosing c large enough, we can certainly force $g''(x) > 0$ on $(-1, 1)$ ($g[x_0, x_1, x_2] \geqslant \frac{1}{2}$, say). Now the function $h(x) \equiv g[x, -1]$ is bounded and monotonic (since $g'' > 0$ on $(-1, 1)$) and hence $\lim_{x \to -1^+} h(x)$ exists, so that $g'(-1)$ exists, since $g \in C[-1, 1]$. Similarly, $g'(1)$ exists. But we also get that $g'(x) = g[x, x]$ is bounded and monotonic on $(-1, 1)$, and hence $\lim_{x \to -1^+} g'(x)$ and $\lim_{x \to 1^-} g'(x)$ must exist. Then g' must be in $C[-1, 1]$, and hence f' exists on $[-1, 1]$.

Now we just proceed inductively. For $n = 5$ (choosing c larger, perhaps, as we go along) we can force $g'''(x) > 0$ on $(-1, 1)$. Then $g'[x, -1]$ is bounded and monotonic (just use the Mean Value Theorem) and hence $g''(-1)$ (and similarly $g''(1)$) exists with $g'' \in C[-1, 1]$. Proceeding, we see that $f^{(n)}$ exists on $[-1, 1]$ for any given positive integer n.

Remark. It can be shown that there exists a function $f \in C^\infty[-1, 1]$ such that neither f nor $-f$ satisfies the hypothesis of Theorem 2. This follows from [HR1], where an $f \in C^\infty[-1, 1]$ is constructed so that $\mathscr{L}(\mathscr{L}(f)) = \{\text{all polynomials}\}$.

Question. Must the f in Theorem 2 *be analytic on* $[-1, 1]$?

COROLLARY 1. *If* $f \in \mathrm{TBI}$, *then* $f \in C^\infty[-1, 1]$.

Proof. Follows immediately from Theorem 2. There is also a proof of Corollary 1 that is simpler than that of Theorem 2.

Now that we know that $\mathrm{TBI} \subseteq C^\infty[-1, 1]$, we can use the partial sums of the *Taylor series* to get that f is actually analytic on $[-1, 1]$. (Note that our bound on interpolants with distinct nodes extends easily when the nodes coalesce.)

LEMMA 1. *For any* $f \in \mathrm{TBI}$,

$$\left|\frac{f^{(n)}(x)}{n!}\right| \leqslant \frac{2\, \|f\|_{\mathrm{TBI}}}{(1+|x|)^n}.$$

Proof. Consider for any $x \in I$ the Taylor interpolant

$$s_n(x; c) = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

Then

$$|s_n(x; c) - s_{n-1}(x; c)| \leqslant 2\, \|f\|_{\mathrm{TBI}}$$

so that

$$\frac{|f^{(n)}(c)|}{n!}|x-c|^n \leqslant 2\, \|f\|_{\mathrm{TBI}} \Rightarrow \frac{|f^{(n)}(c)|}{n!} \leqslant \frac{2\, \|f\|_{\mathrm{TBI}}}{(1+|c|)^n}.$$

THEOREM 3. *Let* $E =$ *union of the two discs in the complex plane* $E_1 = \{z\colon |z-1| < 2\}$ *and* $E_2 = \{z\colon |z+1| < 2\}$. *Then if* $f \in \mathrm{TBI}$, f *must extend to be analytic in* E.

Proof. For any $c \in [-1, 1]$,

$$f(x) - s_n(x; c) = \frac{1}{n!}\int_c^x f^{(n+1)}(t)(x-t)^n\, dt$$

(the Integral Form for the Remainder), and thus

$$|f(x) - s_n(x; c)| \leqslant \frac{1}{n!}\, 2\, \|f\|_{\mathrm{TBI}}\, (n+1)! \int_c^x \frac{|(x-t)^n|}{(1+|t|)^n}\, dt$$

$$\leqslant k(n+1)\left.\frac{(x-t)^{n+1}}{n+1}\right|_c^x \qquad \text{or} \qquad k(n+1)\left.\frac{(t-x)^{n+1}}{n+1}\right|_c^x$$

(by Lemma 1) and thus $|f(x) - s_n(x; c)| \leqslant k\, |(x-c)|^{n+1}$. Hence for x close to c, the Taylor series converges to $f(x)$, which implies that f is *analytic* on $[-1, 1]$. For each c, the radius of convergence r_c of the power series expansion about c is $\geqslant 1+|c|$, again by Lemma 1. Thus f is analytic in $\bigcup \{N_c: c \in [-1, 1]\}$, where $N_c = \{z: |z-c| < 1+|c|\}$. But it is easily seen that $\bigcup N_c = E$, and this completes the proof.

Remark. While we have defined TBS for *real-valued* functions, if f is complex-valued then the corresponding definition is obvious (or one can say then that $f \in \mathrm{TBS} \Leftrightarrow \operatorname{Re} f$ and $\operatorname{Im} f$ are in TBS).

LEMMA 2. *For any w on ∂E or outside $\bar{E}$, $f(z) = 1/(w-z) \in \mathrm{TBI}$.*

Proof. Let p be any interpolant to f at $\{x_0, ..., x_n\} \subseteq [-1, 1]$. Then

$$E(x) = f(x) - p(x) = \frac{(x-x_0)\cdots(x-x_n)}{(w-x_0)\cdots(w-x_n)(w-x)}.$$

(This follows easily since $p(x)(w-x)$ interpolates 1 at $\{x_0, ..., x_n\}$, etc.) But for any j we must have $|x-x_j| \leqslant |w-x_j|$ since E contains every disc in $\mathbb{C}$ centered at x_j with radius $|x-x_j|$, $x \in [-1, 1]$.

Remark. It is of considerable interest to determine the precise boundary behavior in E of functions $f \in \mathrm{TBI}$. For example, using Cauchy's formula, it can be shown that $H^1(E) \subseteq \mathrm{TBI}$, where

$$H^1(E) = \{\text{Hardy Space of } H^1 \text{ functions on } E\}$$
$$= \left\{ f \text{ analytic in } E\colon \lim_{\Gamma \to \partial E} \int_\Gamma |f(x)|\, d|x| \right\}$$

exists for any collection of uniformly smooth contours Γ tending to ∂E. However, Lemma 2 shows that $H^1(E) \neq \mathrm{TBI}$ since $1/(3-z) \notin H^1(E)$.

2. SPACE OF TOTALLY DERIVATIVE BOUNDED FUNCTIONS

We now define a Banach space B as follows:

$$f \in B \Leftrightarrow \exists M \text{ such that } \|p^{(j)}\|_{\infty, [-1, 1]} \leqslant M$$

for any $p \in \mathscr{L}(f)$ and any non-negative integer j. For $f \in B$,

$$\|f\|_B \equiv \sup_{\substack{p \in \mathscr{L}(f) \\ j = 0, 1, ...}} \|p^{(j)}\|_{\infty, [-1, 1]}.$$

It is trivial that this defines a norm, and it also follows that B is complete under this norm. We sketch the proof of that fact now. Note that since $f \in \text{TBI}$, f is certainly $\in C^{\infty}[-1, 1]$.

By taking nth derivatives of Taylor interpolants of f, it is easily seen that

$$\|f^{(j)}\|_{\infty, [-1, 1]} \leqslant M \qquad \text{independent of } j \tag{8}$$

and

$$\sup_j \|f^{(j)}\|_{\infty, [-1, 1]} \leqslant \|f\|_B.$$

So if $\{f_n\}$ is Cauchy in B, $\exists f \in C^{\infty}[-1, 1]$ such that $f_n^{(j)} \to f^{(j)}$ uniformly for any j (we really do not need such convergence, though). Then we just proceed as earlier, using the fact that $D^j L(f_n)(x) \to D^j L(f)(x)$ for any $L \in \mathscr{L}(f)$ and any j.

Now it also follows from (8) that

$$\text{Every } f \text{ in } B \text{ is an entire function.} \tag{9}$$

We can also show

THEOREM 4. *B is not an algebra.*

Proof. First, for $f(x) = e^{cx}$, $|c| \leqslant 1$, $f \in B$. This follows since for any $p \in \mathscr{L}(f)$, $p^{(j)} \in \mathscr{L}(f^{(j)})$ and hence

$$|f^{(j)}(x) - p^{(j)}(x)| = \left| (x - t_0) \cdots (x - t_{n-j}) \frac{f^{(n+1)}(\xi)}{(n-j+1)!} \right|$$
$$\leqslant e^{|c|} \frac{2^{(n-j+1)}}{(n-j+1)!} |c|^{n+1},$$

which clearly remains bounded, independently of n and j.

If $c > 1$, however, then e^{cx} clearly does not satisfy (8) and hence is not in B.

Question 1. Is B a familiar space of entire functions? In particular, is $B = \{\text{entire functions of type } \leqslant 1\}$, with the B norm equivalent to $\sup_j \|f^{(j)}\|_{\infty, I}$?

Question 2. We can define a similar space A by the requirement $\|p\|_{\infty, I} \leqslant M$, where p is any Lagrange interpolant to any derivative of f. By the Mean Value Theorem, $A \subseteq B$. Is $A = B$ (setwise), with the norms equivalent? (The norm on A is obvious.)

3. Totally Positive Functions on I and Related Topics

Our first result follows directly from Theorem 2.

THEOREM 5. *If* $f \in \mathrm{TPI}$, *then* $f \in C^\infty[-1, 1]$.

THEOREM 6. $f(x) = 1/(b-x) \in \mathrm{TPI} \Leftrightarrow b \geqslant 3$.

Proof. Clearly we must have $b > 1$. As noted earlier,

$$E(x) = f(x) - p(x) = \frac{(x-x_0)\cdots(x-x_n)}{(b-x_0)\cdots(b-x_n)(b-x)},$$

where $p = L(f; x_0, ..., x_n)$. Also p is positive on $[-1, 1]$,

$$\Leftrightarrow E(x) < f(x) \text{ on } [-1, 1] \Leftrightarrow \frac{(x-x_0)\cdots(x-x_n)}{(b-x_0)\cdots(b-x_n)} < 1 \text{ on } [-1, 1].$$

But if $1 < b < 3$ and n is odd then choose $x_0 = \cdots = x_n = 1$ and $x = -1 \Rightarrow E(-1) > f(-1)$, and a small perturbation gives an interpolant at distinct points which is negative at -1.

If $b > 3$ the result is trivial. If $b = 3$, then when $x_0, ..., x_n$ are *distinct* $(x-x_0)\cdots(x-x_n) < (b-x_0)\cdots(b-x_n)$.

It was noted in the Introduction that if $f \in \mathrm{TBI}$, then $f + M \in \mathrm{TPI}$ for large enough M. It is unclear, however, what the exact connection is. It is plausible that the answer to the following question is yes.

Question. Is $\mathrm{TPI} \subseteq \mathrm{TBI}$?

We can prove a result like this if we assume that *all* the derivatives are totally positive. We find it convenient for now to work on $[0, 1] = J$.

THEOREM 7. *Suppose* $f \in C^\infty[0, 1]$ *and* $f^{(j)} \in \mathrm{TPJ}$ *for all* j. *Then* $f^{(j)} \in \mathrm{TBJ}$, $\forall j$.

Proof. Clearly we must have $f^{(j)} > 0$ on J, $\forall j$. Now if $p \in \mathscr{L}(f^{(j)})$, then

$$E(1) \equiv f^{(j)}(1) - p(1) = (1-x_0)\cdots(1-x_n) f^{(n+j+1)}(\xi)/(n+1)!$$
$$\geqslant 0 \Rightarrow p(1) \leqslant f^{(j)}(1).$$

Also $p(0) \geqslant 0$. Since p is monotone on $[0, 1]$ (because $p' > 0$ since $p' \in \mathscr{L}(f^{(j+1)})$, we must have $\|p\|_\infty = p(1) \leqslant f^{(j)}(1)$ for all interpolants p to $f^{(j)}$. Hence $f^{(j)} \in \mathrm{TBJ}$.

Remark. There *are* functions f such that $f^{(j)} \in \mathrm{TPJ}$, $\forall j$. For example, if g belongs to the space A mentioned earlier (uniform bound on interpolants to any derivative, modified for $[0, 1]$), then $g + Me^x$ will work for large M, since it can easily be shown that e^x (and hence all its derivatives) is in TPJ, with a *positive lower bound* on all the interpolants to e^x. However, if $f^{(j)} \in \text{TBJ}\ \forall j$, we cannot just take $f + Me^x$ to get $f^{(j)} \in \text{TPJ}\ \forall j$.

Interpolants on Unbounded Intervals

THEOREM 8. *Suppose f is totally positive on $[0, \infty)$. Then $f(x) = ax + b$ for some constants a and b.*

First, we state the following lemma, whose simple proof we leave to the reader.

LEMMA 3. *Suppose all linear interpolants to f are positive on $[0, \infty)$, where the nodes are also from $[0, \infty)$. Then $f(x)/x$ must be decreasing on $(0, \infty)$.*

Proof of Theorem 8. By Lemma 3, $f(x) = O(x)$ as $x \to \infty$. Now suppose f is *not linear*, and choose points $\{x_0, x_1, x_2\} \subseteq [0, \infty)$ such that $f[x_0, x_1, x_2] \neq 0$. Let $p(x) = L(f; x_0, x_1, x_2)$, so that p has degree $= 2$ since its leading coefficient is $f[x_0, x_1, x_2]$. Let

$$E(x) = f(x) - p(x) = (x - x_0)(x - x_1)(x - x_2) f[x, x_0, x_1, x_2].$$

Now since all *third*-degree interpolants to f are positive on $[0, \infty)$, we must have $f[x_3, x_0, x_1, x_2] \geqslant 0$ for all points $0 \leqslant x_0 < x_1 < x_2 < x_3 < \infty$. Hence $E(x) \geqslant 0$ for all $x \geqslant x_2$. But $f(x) - p(x) \to -\infty$ as $x \to \infty$ since $f(x) = O(x)$ and p has degree $= 2$. This contradiction implies that f must be linear.

Remarks. (i) In proving Theorem 8 we really only used the fact that all the interpolants of degree 1, 2, and 3 are positive on $[0, \infty)$.

(ii) A necessary condition for f to belong to TPJ is that $f(x)/x$ be decreasing on $(0, 1)$. Hence $f(x) = (x + \varepsilon)^2 \notin \text{TPJ}$ for small $\varepsilon > 0$. But $f(x) = x + \varepsilon$ is in TPJ. Hence TPJ is *not* an algebra!

We now prove a result similar to Theorem 8 for interpolants on the real line R, where we assume the degree is even, of course.

THEOREM 9. *Suppose that all interpolants of even degree to f with nodes in R are positive on R. Then $f(x) = ax^2 + bx + c$ for some a, b, c.*

Proof. It suffices to assume that f is *even* on R. (If $f \in \text{TPR}$, then

$$g(x) = \frac{f(x) + f(-x)}{2} \in \text{TPR}.$$

We will have that $g(x)$ is quadratic. Let $h(x) = (f(x) - f(-x))/2$ be the odd part of f. Since $f''''(x) \geqslant 0$ for all $x \in \mathbb{R}$, we have $h''''(x) \geqslant 0$ for all $x \in \mathbb{R}$, and thus h is at worst a cubic. But h cannot contain an x^3 term since then f would, yet f is non-negative on $\mathbb{R}$. Hence f must also be quadratic.)

First we claim

$$f(x) = O(x^2) \qquad \text{as } x \to \infty. \tag{10}$$

To prove (10), consider $p(x) = L(f: -x_1, x_1, x_2)$ with $0 < x_1 < x_2$ so that

$$\begin{aligned} p(x) &= f(-x_1) + f[-x_1, x_1](x + x_1) + f[-x_1, x_1, x_2](x^2 - x_1^2) \\ &= f(x_1) + f[-x_1, x_1, x_2](x^2 - x_1^2), \end{aligned}$$

since f is even. Then

$$p(0) = f(x_1) - x_1^2 f[-x_1, x_1, x_2] > 0$$

(again f even $\Rightarrow f[-x_1, x_1, x_2] = f[x_2, x_1]/(x_1 + x_2)$)

$$\Rightarrow f(x_1) \geqslant \frac{x_1^2}{x_2 + x_1} f[x_2, x_1],$$

and hence $f(x_1)/x_1^2 \geqslant f(x_2)/x_2^2$ so that $f(x)/x^2$ is decreasing on $(0, \infty)$, and (10) follows immediately.

It is clear that all the even-order divided differences of f must be non-negative and hence $f \in C^\infty(R)$ by [BW] as earlier. (To apply the result in [BW], f must be continuous, but convex functions on R are continuous on R.)

Since f is even, $f^{(n)}(x)$ is $\geqslant 0$ for $x \geqslant 0$ and $\leqslant 0$ for $x \leqslant 0$, whenever n is *odd*, and in particular when $n = 3$. Now assume f is *not quadratic*. Then we can choose points $\{x_0, x_1, x_2, x_3\}$ in $(0, \infty)$ such that $f[x_0, ..., x_3] \neq 0$. Let $p(x) = L(f; x_0, ..., x_3) \Rightarrow \deg p = 3 \Rightarrow E(x) = f(x) - p(x) \to -\infty$ as $x \to \infty$ by (10). But $E(x) = (x - x_0) \cdots (x - x_3) f^{(4)}(\xi)/4!$, where $\xi \geqslant 0$, so that $E(x) \geqslant 0$ for x large—a contradiction.

Remark. It can be seen that we only used the positivity of the quadratic and fourth-degree interpolants. (We need the latter to force $f^{(4)} > 0$ so that $\Rightarrow f^{(3)}$ is increasing on $[0, \infty)$, etc.) What if we just consider interpolants of degree 2?

It is true that there are non-quadratics f such that every second-order *Taylor* interpolant is positive on R (by looking at the discriminant of $f(x_0) + f'(x_0)(x - x_0) + f''(x_0)(x - x_0)^2/2$, we get the condition $(f')^2 < 2ff''$ on R). Thus $f(x) = e^x$ is such a function. However, it follows easily that not *every* quadratic interpolant to e^x is positive on R, since e^x dominates all quadratics at $+\infty$.

4. Open Questions and Future Research

In addition to some of the questions already listed in this paper, there are many others that come to mind. We list just a few. A space closely related to TBI is TCI, the space of totally convergent functions on I. $f \in \text{TCI}$ if any sequence $\{p_n\} \subseteq \mathscr{L}(f)$ of polynomials of increasing degree converges to f uniformly on I. It can be shown that

$$\text{TCI is a closed subspace of TBI,} \tag{11}$$

$$\text{TCI} = \{\text{closure of the polynomials in the TBI norm}\}. \tag{12}$$

(For related work on TBD and TCD, D the unit disc, see [HR2]. The flavor of that paper is generally different, however.)

Problem 1. Is $\text{TCI} \approx c_0$ and $\text{TBI} \approx l^\infty$, where $\approx$ denotes isometric isomorphism?

Problem 2. Is $(\text{TCI})^{**} = \text{TBI}$?

Problem 3. We have seen that

$$f \in \text{TBI} \Rightarrow \frac{|f^{(n)}(c)|}{n!} \leqslant \frac{K}{(1+|c|)^n} \qquad \text{for} \quad c \in I.$$

Is this condition *sufficient*?

Problem 4. Is the above condition sufficient for the partial sums of the *Taylor series* to be uniformly bounded (called Taylor bounded)? (Using the Integral Form of the Remainder, this does not seem easy and perhaps involves the solution of some extremal problem.)

Closely related to Problem 4 is:

Problem 5. Does Taylor bounded $\Rightarrow$ totally bounded?

Problem 6. Does Taylor positive $\Rightarrow$ totally positive?

Problem 7. Is TBI non-separable? (See [HR2] for a related result, if I is replaced by D, the unit disc.)

A whole class of problems arises as follows:

Project 1. Analyze the questions in this paper for *other norms*—such as $L^p[-1, 1]$, $\text{BMO}[-1, 1]$, etc.

Project 2. Choose the interpolating points from one set S_1 and the sup norm on another set S_2, and then proceed.

Problem 8. Analyze all of the above where the interpolating points are *equi-spaced* on $[-1, 1]$. What does the corresponding Banach space look like?

Other Notions

Problem 9. What properties of $\mathscr{L}(f)$ imply that f is *continuous*? (It is true that $f \in C[-1, 1] \Leftrightarrow$ *some sequence* from $\mathscr{L}(f)$ converges uniformly to f on I. But this doesn't really involve just the *intrinsic properties of* $\mathscr{L}(f)$ *itself*, without any reference to f.)

Problem 10. Suppose every interpolant to f has all its zeroes in I or all real zeroes. What can be said about f? (For related notions on domains in the plane, see [HR3].)

Problem 11. What are the extreme points of the cone of totally non-negative functions on $[-1, 1]$?

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